Third– and Higher–Price Auctions[1]

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Abstract

This paper solves the equilibrium bid functions of third– and higher–price auctions for a large class of distribution functions of bidders' valuations, assuming the symmetric independent private values framework, and risk neutrality. In all these auctions, equilibrium bids *exceed* bidders' valuations, and bidders raise their bids when one moves to a higher price auction, and lower bids when the number of bidders is increased.

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1 Introduction

In the analysis of independent private value auctions it is a common exercise to compare equilibrium bidding in first– and second–price auctions. While truthful bidding is a (weakly) dominant strategy in second–price auctions, bidders necessarily "shade" their bids below their valuations when they participate in a first–price auction. Therefore, switching from a second– to a first–price auction leads to lower bidding. However, when the number of bidders is increased, the participants of a first–price auction raise their bids, which in turn reduces the gap between the two equilibrium bidding rules.

From first– and second–price auctions one may extrapolate, and design third– and higher–price auctions, where the winner pays the third–highest bid, or more generally the k–th highest bid, for $k > 2$. This raises the question: can one also extrapolate how equilibrium bidding changes when one moves from second– to third– and higher–price auctions, and when the number of bidders is increased?

The present paper solves equilibrium bid functions of third– and higher–price auctions for a large class of distribution functions of bidders valuations, assuming the symmetric independent private values framework and risk neutrality. The solutions imply the following general properties of the k–price auction, for $k > 2$,

1. equilibrium bids *exceed* bidders' valuations (the opposite of "shading")
2. moving to a higher–price auction leads to higher bidding (equilibrium bids increase in k)
3. equilibrium bids diminish when the number of bidders is increased
4. moving to a higher–price auction tends to increase the variance of the equilibrium price.

These properties delineate a general pattern of how equilibrium bidding rules change as we move from first– to second–, and higher–price auction, all the way to the n–th price auction, where the winner pays "only" the lowest bid, and they may explain the predominance of first–price auctions.

Third–price auctions were considered for the first time by Kagel and Levin [1993], who solved the equilibrium bid function assuming uniformly distributed valuations, and who identified some further general properties, but did not give a general solution of equilibrium bid functions.

Third–price auctions have been useful to test the predictive power of auction theory in laboratory experiments (see Kagel and Levin [1993]).

2 Preliminaries

The solution builds upon some generally known results that are briefly summarized in this section, after stating basic assumptions and notation. It shows how the indirect solution method introduced by Riley and Samuelson [1981] is useful to solve even relatively complicated auctions.

Assumptions Consider a k–price auction, where a single unit is sold to $n \geq k \geq 3$ risk neutral bidders. This auction is characterized by three rules: 1) the item is awarded to the highest bidder (ties are handled by some allocation rule); 2) the winner pays the k–th highest bid; 3) only the winner pays.

The analysis assumes the symmetric independent private values framework. From the seller's perspective (and that of rival bidders), buyers' valuations $V_1, \ldots, V_n$ are continuous, iid, random variables, with distribution function (cdf) $F(v)$, and density function (pds) $f(v)$, on the support $[0, \bar{v}]$, $\bar{v} > 0$.

Bidders' strategy is their bid function $b_k(v) : [0, \bar{v}] \to \mathbb{R}_+$, and participation rule $\xi_k(v) : [0, \bar{v}] \to \{0, 1\}$. We characterize symmetric equilibria, where each bidder bids according to the same bid function $b_k^*(v)$, and assume that all bidders have an incentive to participate, which is always the case when the item is unconditionally awarded to the highest bidder.

Bidders' probability of winning is denoted by ρ, and their (ex ante) expected payment by $\mathcal{E}$. To avoid confusion, random valuations are written in capital and realizations in lowercase letters.

Order statistics Order statistics are a useful tool for analyzing auctions. Arrange the $n \geq 2$ iid random valuations $V_1, V_2, \ldots, V_n$ in ascending order of magnitude, and write them as $V_{(1)} \leq V_{(2)} \leq \ldots \leq V_{(n)}$. The random variable $V_{(r)}$ is called the r–th "order statistic" ($r = 1, 2, \ldots, n$); $V_{(1)}$ is the "lowest", and $V_{(n)}$ the "highest valuation".

Order statistics are necessarily dependent, because of the inequality relationship between them, and they are not identically distributed, even when the underlying $V's$ are iid random variables.

The probability density (pds) of the r-th order statistic $V_{(r)}$ is $f_{V_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} (1 - F(x))^{n-r} f(x)$. And the conditional pds of $V_{(r)}$, given $V_{(s)} = v$ ($s > r$), is just the pds of $V_{(r)}$ in the smaller sample size $(s-1)$, drawn from the parent distribution truncated on the right at v (see Theorem 2.7 in David [1970]). Therefore, for $x < v$,

$$f_{V_{(r)}|V_{(s)}=v}(x) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{f(x)F(x)^{r-1} (F(v) - F(x))^{s-r-1}}{F(v)^{s-1}}. \tag{2.1}$$

This result will be used repeatedly.

Basic results All auctions that award the item to the highest bidder are payoff equivalent, give rise to the same allocation rule, and exhibit strict monotone increasing equilibrium bid functions (see Riley and Samuelson [1981] and Myerson [1981]). In particular, they all share the same equilibrium probability of winning, and the same equilibrium expected payments, which are summarized below.

Proposition 1 (Riley/Samuelson (1981)) *Consider auctions that award the item to the highest bidder. In equilibrium, the probability of winning, $\rho^*(v)$, and (ex ante) expected payments $\mathcal{E}^*(v)$ are*

$$\rho^*(v) = \Pr\{V_{(n-1)} < v\} = F(v)^{n-1} \tag{2.2}$$

$$\mathcal{E}^*(v) = vF(v)^{n-1} - \int_0^v F(x)^{n-1}dx \tag{2.3}$$

The equilibrium expected payment has an appealing interpretation:

Proposition 2 (Expected payment) *The conditional equilibrium expected payment, conditional upon winning, is equal to the conditional expected value of the second highest valuation: $E[V_{(n-1)} \mid V_{(n)} = v]$,*

$$\begin{aligned} \mathcal{E}^*(v) &= \rho^*(v;n)E[V_{(n-1)} \mid V_{(n)} = v] && (2.4) \\ &= \int_0^v x(n-1)F(x)^{n-2}f(x)dx. && (2.5) \end{aligned}$$

Proof By (2.3) one has

$$\begin{aligned} \mathcal{E}^*(v) &= xF(x)^{n-1}\Big|_{x=0}^{x=v} - \int_0^v F(x)^{n-1}dx && (2.6) \\ &= \int_0^v x(n-1)F(x)^{n-2}f(x)dx. && (2.7) \end{aligned}$$

Utilizing (2.1), it follows immediately that the latter is equal to $\rho^*(v;n)E[V_{(n-1)} \mid V_{(n)} = v] := \int_0^v x f_{V_{(n-1)}|V_{(n)}=v}(x)dx$. ■

For simple but complete proofs of these and other basic results of auction theory consult the extensive survey by Wolfstetter [1996].

3 Equilibrium Bid Functions

Proposition 3 (Third–price auction) *Consider the third–price auction, where the highest bidder wins, and pays "only" the third highest bid, and assume the probability distribution function F is log–concave. Then, the equilibrium bid function is*

$$b_3^*(v) = v + \frac{F(v)}{(n-2)f(v)} \quad \textit{(3–rd price auction)}. \tag{3.1}$$

Proof In a third–price auction one has

$$\begin{aligned}
\mathcal{E}^*(v) &= \rho^*(v;n)E[b_3^*(V_{(n-2)}) \mid V_{(n)} = v] \\
&= \rho^*(v;n)\int_0^v b_3^*(x) f_{V_{(n-2)}|V_{(n)}=v}(x)dx \\
&= (n-1)(n-2)\int_0^v b_3^*(x)F(x)^{n-3}(F(v)-F(x))f(x)dx. \qquad (3.2)
\end{aligned}$$

Use (2.5) from Proposition 2, and one has for all v

$$\int_0^v xF(x)^{n-2}f(x)dx = (n-2)\int_0^v b_3^*(x)F(x)^{n-3}(F(v)-F(x))f(x)dx. \qquad (3.3)$$

Twice differentiate this identity with respect to v, rearrange, and one obtains the asserted equilibrium bid function.

This derivation assumes that the equilibrium bid function is strict monotone increasing in v. The assumed log–concavity[1] of F entails that $\frac{F(v)}{f(v)}$ is increasing which assures that the equilibrium bid function is indeed increasing. ■

Proposition 4 (Generalization) *Consider the k–price auction, where the highest bidder wins, and pays "only" the k–th highest bid, $k \in \{2, 3, \ldots, n\}$, and assume F is log–concave. Then, the equilibrium bid function is*

$$b_k^*(v) = v + \frac{k-2}{n-k+1}\,\frac{F(v)}{f(v)} \quad (k\text{–}\textit{price auction}). \qquad (3.4)$$

Proof In a k–price auction one has $\mathcal{E}^*(v) = \rho^*(v;n)E[b_k^*(V_{(n-k+1)}) \mid V_{(n)} = v]$. By a procedure similar to the above, one arrives at the following identity in v

$$\int_0^v x(n-1)F(x)^{n-2}f(x)dx = \qquad (3.5)$$

$$\int_0^v b_k^*(x)\frac{(n-1)!}{(n-k)!(k-2)!}f(x)F(x)^{n-k}\left(F(v)-F(x)\right)^{k-2}dx. \qquad (3.6)$$

Differentiate this identity $(k-1)$–times, dividing by $f(v)$ after each round of differentiation (including the last). Then, one obtains for the right–hand side (RHS) of (3.5)

$$\text{RHS} = b_k^*(v)\frac{(n-1)!}{(n-k)!}F(v)^{n-k}. \qquad (3.7)$$

[1]A function F is log–concave if $\ln(F)$ is concave. Log concavity is frequently assumed in information economics, and it is assured by all standard distribution functions; see Bagnoli and Bergstrom [1989].

To apply the same procedure to the left–hand side (LHS) of (3.5) is more tedious. Using the product rule of differentiation $(g(x)h(x))^{(n)} = \sum_0^n \binom{n}{r} g^{(r)}(x)h^{(n-r)}(x)$, and the rule for the r–th derivative of the polynomial $\psi(x) := x^n$, $\psi^{(r)}(x) = r!\binom{n}{r}x^{n-r}$, one obtains, after some rearranging,

$$\text{LHS} = \frac{(n-1)!}{(n-k)!}F(v)^{n-k}\left(v + \frac{k-2}{n-k+1}\frac{F(v)}{f(v)}\right). \tag{3.8}$$

Equate RHS and LHS, and one has the asserted bid function. The assumed montonicity is again assured by the log–concavity of F. ∎

From these results, the properties of the k–price auction postulated in the introduction follow immediately.

Example 1 (Kagel/Levin (1993)) *If valuations are uniformly distributed on the support* $[0,1]$*, one has* $b_k^*(v) = \frac{n-1}{n-k+1}v$*, as in Kagel and Levin. Incidentally, in this case* $b_k^*(v)$ *covers all considered auctions — from the first–price auction, where bidders bid a fraction of their valuation,* $b_1^*(v) = \frac{n-1}{n}v$*, to the* n*–th price auction, where bidders bid* $(n-1)$ *times their valuation,* $b_n^*(v) = (n-1)v$*.*

Remark 1 *In the case of a uniform distribution of valuations it is easy to see that the variance of the random equilibrium price* P_k *tends to increase in* k*, since*

$$\text{Var}(P_k) = \text{Var}\left(b_k^*(V_{(n-k+1)})\right) = \frac{k(n-1)}{(n-k+1)(n+1)^2(n+2)^2}. \tag{3.9}$$

This suggests that the seller finds second–, third, and higher–price auctions unappealing in terms of risk, which may explain the prevalence of first–price auctions.

4 Conclusions

Third– and higher price auctions have four striking properties: 1) bids are higher than the own valuation; 2) equilibrium bids increase when one moves to a higher–price auction; 3) equilibrium bids diminish as the number of bidders is increased; and 4), the riskiness of the random equilibrium price tends to increase as one moves to higher–price auctions.

Once one has figured out why it pays to "speculate", and bid higher than one's own valuation, it is easy to interpret the third property. Just keep in mind that a rational bidder may get "burned", and suffer a loss, because the k–th highest bid is above the own valuation. As the number of bidders is increased, it becomes more likely that the k–th highest bid is in close vicinity to the own valuation. Therefore, it makes sense to bid more conservatively when the number of bidders is increased.

Finally, the fourth property suggests that a risk averse seller should always prefer lower order k–price auctions, and should most prefer the first–price auction. While the second–price auction is always appealing because of its overwhelming strategic simplicity, third– and higher–price auctions are strategically just as complicated as the first–price auction, but in addition expose the seller to unnecessary risk.

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